Demand

Economics 100A
Winter 2021

1 Overview

Let's once again restate the utility maximization problem:

$$maxu(x, y)$$
$$s.t.p_x x + p_y y = I$$

As you have seen in class, when we solve this problem for general parameters, we find the demand function. Well, as it turns out, this is called the **Marshallian demand**. We tend to write this as a function of prices and income:

$$x^*(p_x, p_y, I)$$

The choice variable here is quantity demanded and the parameters of the demand function are simply prices and income. Turns out if we change individual parameters, we can see some cool properties of demand functions. These notes do just that.

From the outset, I will try to make my notation as clear as possible. I will write functions in terms of their given parameters and the parameter we change. Also, because we fancy ourselves artists of some variety, we will have a lot of graphical interpretations. This means that we have to be careful about the difference between dependent and independent variables. Prices always go on the vertical (y) axis whereas quantities always go on the horizontal axis (except in the case when we graph quantities of x and y together, then those look like our standard graphs).

Let's summarize for sake of clarity:

1. Solve for the consumer's demand as a function of prices and income. In other words, $x(p_x, p_y, I)$.
2. Change *one* parameter of interest while holding all else constant
3. Re-solve the consumer utility maximization problem
4. Observe how x_i changes as the other parameters change. This should give you a new point on the demand curve
5. If you do this for many values of p_x, p_y, and I, you will get an **offer curve!**

2 Own-Price

It is always useful to have some end goal in mind when doing economics. This helps shape the method of our analysis, as well as guide our interpretation of our results. In the case of own-price demand, we ask: "What is the relationship between the price of a good and its demand?" This helps tremendously in a few ways. Firstly, it allows us to isolate what variable we want to see change: own price. When we look at own-price demand, we are seeing how changes in a good's price changes the demand for this good.

So, mathematically, what exactly are we doing? We are plotting $x(p_x, p_y, I)$ where I and p_y are held fixed. Remember, since this is economics, we must make ridiculous assumptions about the world, including that other prices and income do not change. In any case, when we plot x, we are getting a *demand curve*. How amazing is that?! You have been studying and drawing these things for ages, and now you get to see the foundations of them. The inverse of this curve also has an interesting name. Can you guess it? If you guessed *inverse demand*, then you are correct! Economists have no imagination except for their silly assumptions, it seems.

Let's do a simple example using Cobb-Douglas preferences.

2.1 Cobb-Douglas Demand

I am making a number of short-cuts because at this point I hope you are comfortable with constrained optimization. Recall that the demand for a Cobb-Douglas function can be written as follows:

$$x^*(p_x, I) = x(p_x, p_y, I) = \frac{\alpha}{\alpha + \beta} \cdot \frac{I}{p_x}$$

We know that demand for good x does not rely on the price of y. What else do we see? Since the price of x is in the denominator, we can also see that as the price of x increases, the demand for good x decreases. This is just the law of demand! How splendid. But we are interested in plotting this relationship. And since prices are always graphed on the vertical axis, we have to solve for this function in terms of p (see the math review if you are not comfortable with this).

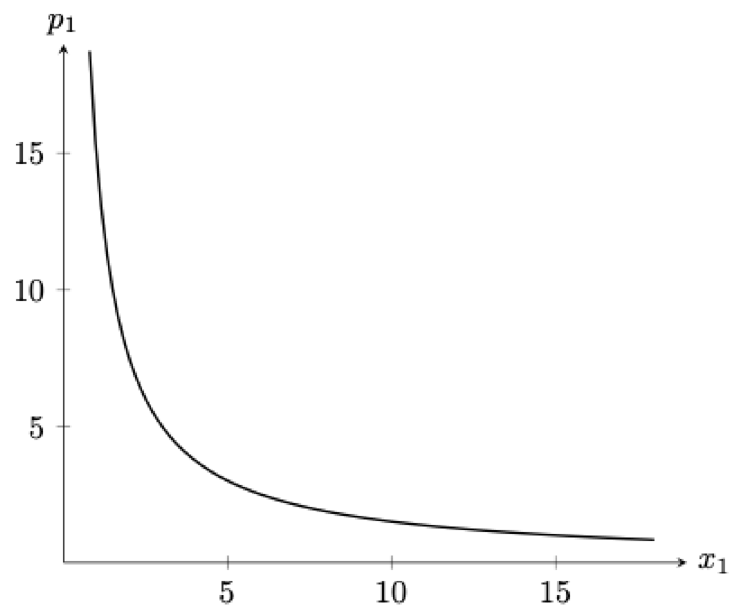
Our function should look like this:

$$p_x(x, I) = p_x(x, p_y, I) = \frac{\alpha}{\alpha + \beta} \cdot \frac{I}{x}$$

Not a whole lot changes. I will leave the algebra to you as an exercise. If you want to see it, come to office hours. So, now that we have that, we can finally plot this thing. Let me give you some nice parameters so we can do that. Suppose that $\alpha = .75$, $\beta = .25$, $p_x = 3$, $p_y = 1$, and $I = 20$. If we plug these into our own-price demand for x, we get:

$$x^*(p_x, 1, 20) = \frac{3}{4} \cdot \frac{15}{p_x}$$

$$p_x(x^*, 1, 20) = \frac{15}{x^*}$$



These are the demand curves for x^*, but we could just as easily solve this for y^*. Using the shortcut yields:

$$y^*(p_y, 3, 20) = \frac{1}{4} \cdot \frac{20}{p_y}$$

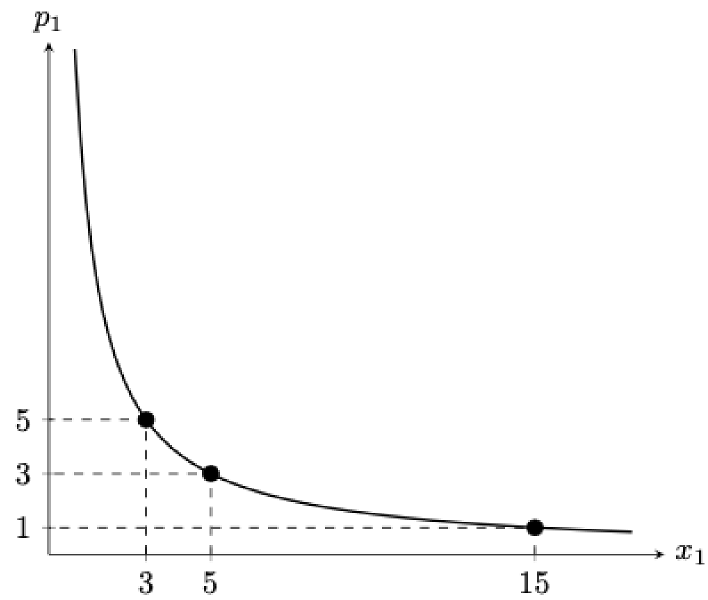
$$p_y(y^*, 3, 20) = \frac{5}{p_y}$$

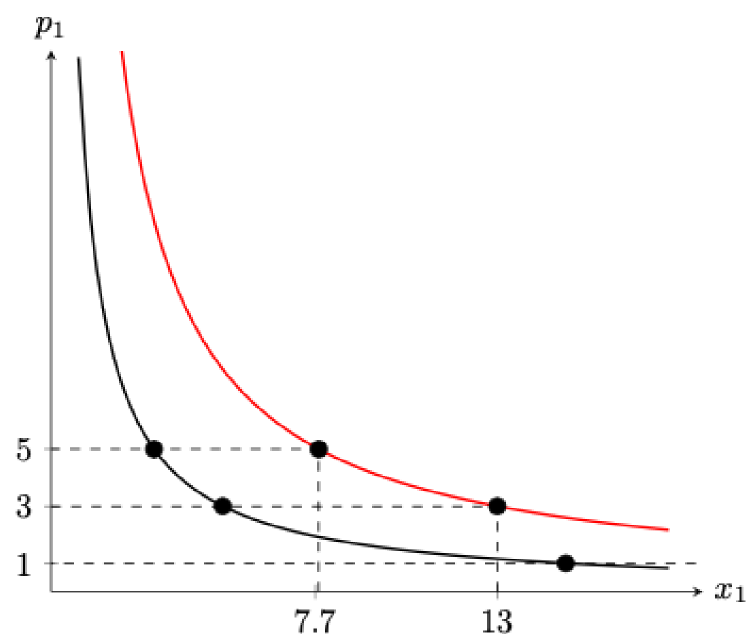
Let's take a moment to see if these demand functions make sense in the first place. Notice how we only have this relationship between prices and optimal quantity of either good. Just by looking at our functions, we see that as the price of x or y increases, we will have a smaller value for x^* and y^*. This is really just the law of demand! As prices increase, the consumer demands fewer of that good.

Let's try graphing this thing! If we plot the demand curve for x, we see a standard, downward-sloping demand curve. However, we must always remember to include p_x on the vertical axis. So, technically, this is an *inverse demand* curve.

Now, what happens when the price of x changes? What happens to demand if p_x=1? What if p_x=3? Or even $p_x = 5$? All we have to do is move along the demand curve to see the different quantities demanded at each different price! As we can see in the diagram, at a $p_x = 5$, the consumer consumes 3 x. Likewise, at a $p_x = 1$, the consumer consumes 15 x. We can verify this algebraically using our demand function above.

Next, suppose we would like to consider a change in income. Suppose that I increases to 52. What happens to demand when income changes? Well, as you should recall, there is an overall shift in the demand curve. As we see, the quantity demanded increases at each and every price level! This





gives us an outward shift in the demand curve. We can also see this using our demand function:

$$x^*(p_x, 1, 52) = \frac{3}{4} \cdot \frac{52}{p_x}$$

$$x^*(p_x, 1, 52) = \frac{39}{p_x}$$

Above, I graph the new curve in red.

As previously stated, the at each price level there is a corresponding higher level of demand.

Let's now look at the price offer curve.[1] Recall that for the offer curve, we must satisfy the following condition:

$$(x(p_x, p_y, I), y(p_x, p_y, I)) = (\frac{\alpha}{\alpha+\beta} \cdot \frac{I}{p_x}, \frac{\alpha}{\alpha+\beta} \cdot \frac{I}{p_y})$$

Since we already have the exponents (alpha and beta), prices and income, we can just rewrite the expression above as:

$$(x(p_x, p_y, I), y(p_x, p_y, I)) = (\frac{15}{p_x}, 5)$$

Since these preferences are Cobb-Douglas, we should not be surprised to see that we have a constant value for y when we consider changes to the price of x. Now, let's be professional and graph this the "proper" way. Our Cobb-Douglas tangency condition was:

$$y = \frac{\beta}{\alpha} \cdot \frac{p_x}{p_y} x$$

Notice that there is a p_x in here. We want to get rid of this, so we are going to plug p_x into $p_x(x, p_y, I) = \frac{\alpha}{\alpha+\beta} \cdot \frac{I}{x}$. This yields:

$$y = \frac{\beta}{\alpha} \cdot \frac{1}{p_y} \cdot (\frac{\alpha}{\alpha+\beta} \cdot \frac{I}{x}) \cdot x$$

$$\therefore y(x, p_y, I) = \frac{\beta}{\alpha+\beta} \cdot \frac{I}{p_2}$$

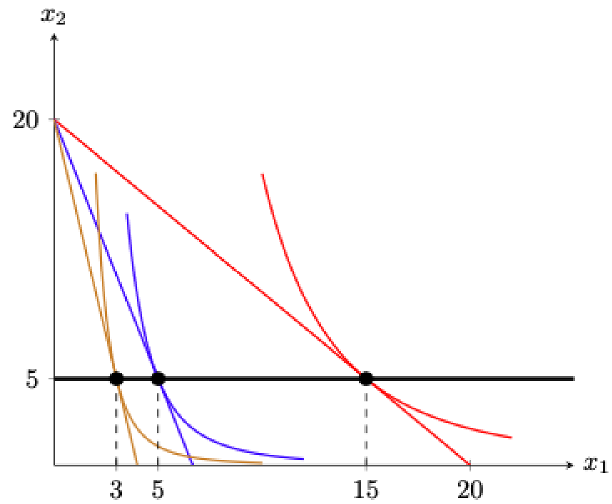
Now, we no longer have p_1 in the equation. This is what we wanted! We also do not have x. This is also expected, since we know that the optimal y is independent of the optimal x. If we plug in the parameters, we get that $y = \frac{1}{4} \cdot \frac{20}{1} = 5$. This is the formula for the price offer curve!

Plotting this price offer curve (in black) and the optimal bundles for $p_x = 1, 3, and 5$ below confirms that each of the tangency points lies on the price offer curve. This creates a locus on all optimal bundles.

3 Cross-Price

As the name implies, here we are interested in how the other price affects our optimal consumption of x and y. We want to fix own-price and income while allowing the price of the other good to vary. This asks: how does the quantity of good i change as the price of good j changes, holding all else constant? Interestingly, this tells us if the goods are compliments or substitutes. This offer curve is called the price offer curve.

[1] I think this is also called the price consumption locus, but I am not sure. I think it's easiest to just call these different offer curves.



3.1 Perfect Compliments

Recall that our demand functions for perfect compliments taking the form $u(x, y) = min\beta x, \alpha y$ can be written in the following general case:

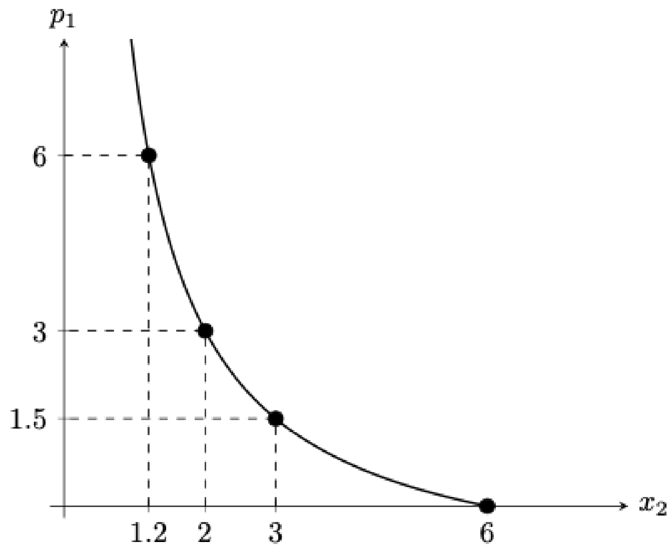
$$x^*(p, I) = \frac{\alpha I}{\alpha p_x + \beta p_y}$$
$$y^*(p, I) = \frac{\beta I}{\alpha p_x + \beta p_y}$$

Let's consider the following parameters: $\alpha = \frac{1}{3}, \beta = \frac{1}{4}, p_x = 3, p_y = 2, I = 12$. Let's just plug these in to find the demand functions of the other price for the above parameters:

$$x^*(p, I) = \frac{\frac{1}{3} \cdot 12}{\frac{1}{3} \cdot 3 + \frac{1}{4} \cdot p_y} = \frac{16}{4 + p_y}$$
$$y^*(p, I) = \frac{\frac{1}{4} \cdot 12}{\frac{1}{3} \cdot p_x + \frac{1}{4} \cdot 2} = \frac{18}{2p_x + 3}$$

Notice how in the general case, since prices are in the denominator, when the price of good i rises, the consumer will consume less good j. This tells us that the good are compliments (surprising considering the name, huh?). In any case, we will need to plot the demand curves for both goods.

Let's start with y as a function of p_x. Again, we start by solving for p_x which should give us the inverse plot $p_x(y, p_y, I)$:



$$y = \frac{\frac{1}{4}I}{\frac{1}{3} + \frac{1}{4}p_y} = \frac{3I}{4p_x + 3p_y}$$

$$\therefore 4p_x + 3p_y = \frac{3I}{y}$$

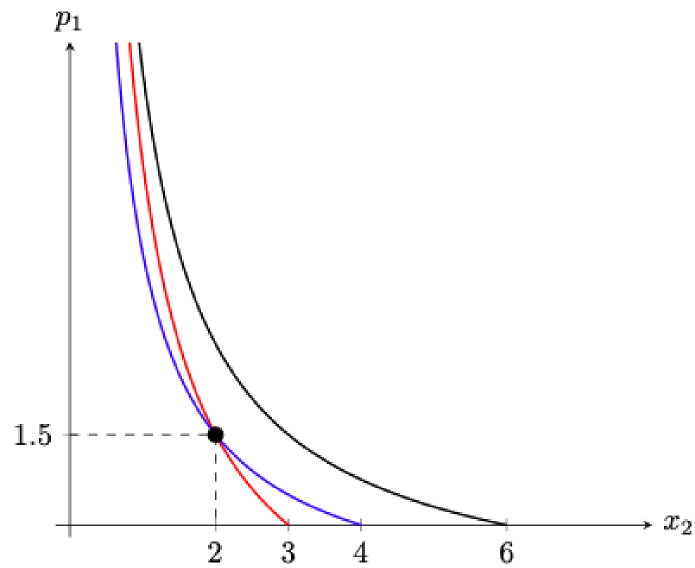
$$p_x = \frac{1}{4}(\frac{3I}{y} - 3p_y)$$

$$p_x(y, p_y, I) = \frac{3I}{4y} - \frac{3}{4}p_y$$

$$\therefore p_x(y, 2, 12) = \frac{9}{y} - \frac{3}{2}$$

I will plot this and comment on a few key features. I will also highlight a few important price levels: $p_x = 1.5$, $p_x = 3$, $p_x = 6$, and $p_x = 0$. This gives us the graph below. Let's note a few key features of this demand curve compared to the Cobb-Douglas demand curve. Firstly, as always, we have price graphed on the vertical axis. Secondly, although we never really consider the case $p_x = 0$ (since when are prices free in the real world?) we do see that the curve can actually touch the x-axis! This is unlike Cobb-Douglas demand curves, which never touch the axes. This is kind of cool, because we can imagine a situation in which the price of the compliment (other good i) is so expensive, the consumer will just consume zero of good j. Right then, the lesson is to check the intercepts!

Now, let's change the price of good y and have it increase to $p_y = 4$. By the law of demand, we expect y to decrease. Now, a more interesting question is how this affects the relationship between y and p_x. We see from the demand curve $p_x(y, p_y, I)$ that a change in p_y only really affects the intercept of the graph – not the slope term. Likewise, if I decreases to, say, 8, we will have a flatter



slope, demand will fall, and the relationship between y and p_x will increase. To see this, we will note the new parameter changes below:

$$y(p_x, 4, 12) = \frac{3 \cdot 12}{4p_x + 3 \cdot 4} = \frac{9}{p_x + 3}$$

$$p_x(y, 4, 12) = \frac{3 \cdot 12}{4y} - \frac{3}{4} \cdot 4 = \frac{9}{y} - 3$$

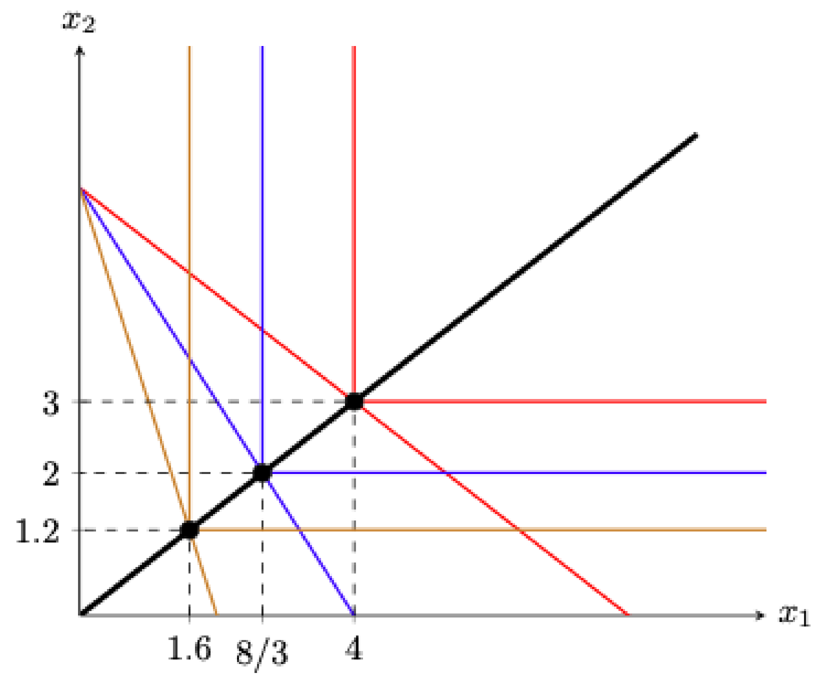
$$y(p_x, 2, 8) = \frac{3 \cdot 8}{4p_x + 3 \cdot 2} = \frac{12}{2p_x + 3}$$

$$p_x(y, 2, 8) = \frac{3 \cdot 8}{4y} - \frac{3}{4} \cdot 2 = \frac{6}{y} - \frac{3}{2}$$

From here, I will plot the change in p_y in red and the change in I in blue. This gives us the following diagram.

Let's finally do the price offer curve. Here, we will do the p_x price offer curve. Let's use the same initial parameter values above: $\alpha = \frac{1}{3}, \beta = \frac{1}{4}, p_x = 3, p_y = 2, I = 12$. Now, our price offer curve is the locus of points satisfying the following conditions:

$$(x(p_x, p_y, I), y(p_x, p_y, I)) = (\frac{\frac{1}{3}I}{\frac{1}{3}p_x + \frac{1}{4}p_y}, \frac{\frac{1}{4}I}{\frac{1}{3}p_x + \frac{1}{4}p_y}) = (\frac{4I}{4p_x + 3p_y}, \frac{3I}{4p_x + 3p_y})$$



$$\therefore (x(p_x, 2, 12), y(p_x, 2, 12)) = (\frac{48}{4p_x + 6}, \frac{36}{4p_x + 6}) = (\frac{24}{2p_x + 3}, \frac{18}{2p_x + 3})$$

This one is a bit harder to visualize than the Cobb-Douglas locus. So, let's think about the formula for the price offer curve. We first need the tangency condition, which for perfect complements occurs at the kinked indifference curve point.

$$y = \frac{\beta}{\alpha}x$$

Notice, then, that this does not have prices (specifically, p_x) in it! We are done! This is exactly the formula for the price offer curve:

$$y(x, p_y, I) = \frac{\beta}{\alpha}x = \frac{1/4}{1/3}x = \frac{3}{4}x$$

Now, we must plot the offer curve as well as the optimal bundles for $p_x = 1.5$, $p_x = 3$, and $p_x = 6$. And now we are done!

3.2 Perfect Substitutes

Let's finally consider a perfect substitutes example: $u(x,y) = \beta x + \alpha y$. We know that their demand function should look like:

$$(x,y) = \begin{cases} (\frac{I}{p_x}, 0) & if \frac{MU_x}{p_x} > \frac{MU_y}{p_y}, \\ (0, \frac{I}{p_y}) & if \frac{MU_x}{p_x} < \frac{MU_y}{p_y}, \\ BudgetLine & if \frac{MU_x}{p_x} = \frac{MU_y}{p_y}. \end{cases}$$

This means that we can write the demand for y as a function of p_x as:

$$y(p_x, p_y, I) = \begin{cases} (\frac{I}{p_x}, 0) & if \frac{MU_x}{p_x} < \frac{MU_y}{p_y}, \\ (0, \frac{I}{p_y}) & if \frac{MU_x}{p_x} > \frac{MU_y}{p_y}, \\ BudgetLine & if \frac{MU_x}{p_x} = \frac{MU_y}{p_y}. \end{cases}$$

Let's set $MU_x = 2$ and $MU_y = 1$. We will consider the following parameter values:

Parameters: (plot in blue)

$$p_x = 1, p_y = 2, I = 10$$
$$\underline{Demand} : y(p_x, 2, 10)$$
$$\begin{cases} 5 & if p_x > 1, \\ \in [0,5] & if p_x = 1, \\ 0 & if p_x < 1. \end{cases}$$

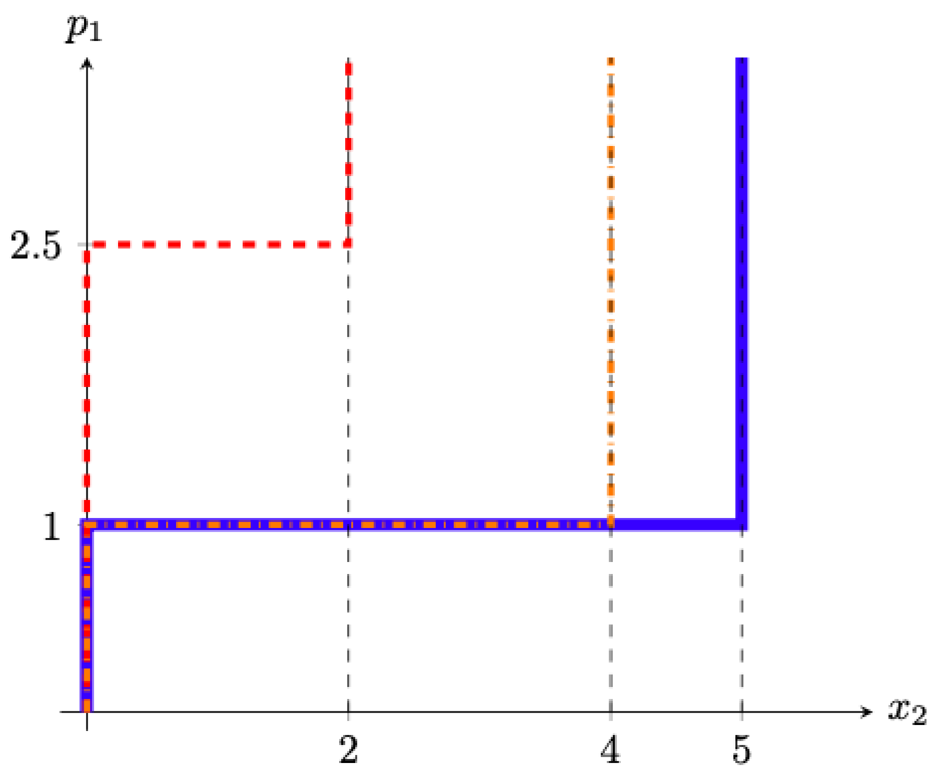
Parameters: (plot in red)

$$p_x = 1, p_y = 5, I = 10$$
$$\underline{Demand} : y(p_x, 5, 10)$$
$$\begin{cases} 2 & if p_x > 2.5, \\ \in [0,2] & if p_x = 2.5, \\ 0 & if p_x < 2.5. \end{cases}$$

Parameters: (plot in orange)

$$p_x = 1, p_y = 2, I = 8$$
$$\underline{Demand} : y(p_x, 2, 8)$$
$$\begin{cases} 8 & if p_x < 4, \\ \in [0,8] & if p_x = 4, \\ 0 & if p_x > 4. \end{cases}$$

We (I) plot these graphs below. The colors are somewhat hard to see, thanks to this stupid Latex package, but you should try drawing this yourself in the meantime. Notice how a change in I does not affect the cutoff value (where there is a straight line) but instead only affects how much good y is purchased. However, changes in p_x do change both how much y is purchased as well as the cutoff.


p_1
2.5
1
x_2
2
4
5

4 Income

I am really tired at this point, but the theme of this section is similar: we want to isolate the consumer's income and plot it against the quantity of good x or y they demand. This asks: how does the quantity of good i change as income changes, holding all else constant? We can call this graph an Engel curve. The offer curve here is called an income offer curve.

4.1 Cobb-Douglas

Returning to our good friends Cobb and Douglas, we will use the same parameters as last time: $\alpha = .75$, $\beta = .25$, $p_x = 3$, $p_y = 1$, and $I = 20$. Our Engel curves are:

$$x(I, p_x, p_y) = \frac{\alpha}{\alpha + \beta} \cdot \frac{I}{p_x}$$
$$\therefore x(I, 3, 1) = \frac{I}{4}$$

$$y(I, p_x, p_y) = \frac{\alpha}{\alpha + \beta} \cdot \frac{I}{p_y}$$
$$\therefore y(I, 3, 1) = \frac{I}{4}$$

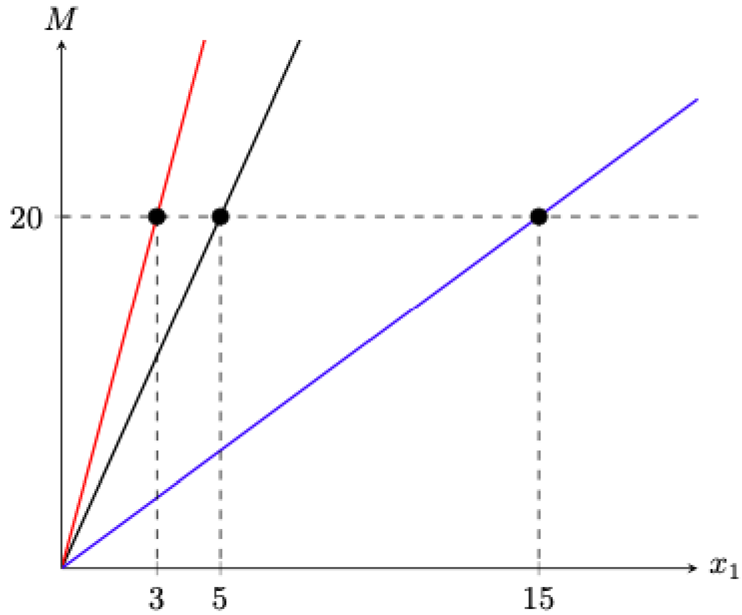
As per usual, we take the inverse of our demand function to plot it. This time, we are plotting:

$$\therefore I(x, 3, 1) = 4x$$
$$\therefore I(y, 3, 1) = 4y$$

I plot the Engel curves below, as well as for $x(I, 5, 1)$ (in red) and $x(I, 1, 2)$ (in blue). At this point, just solve it yourself. You are more than capable and should definitely be able to before the second midterm.

To no one's surprise, Cobb-Douglas preferences give us normal goods! As income increases, we demand more of the goods. We also know that a property of Cobb-Douglas is that the share of income spent on each good stays constant. Since we hold prices fixed, this means that an increase in income will lead to a proportional increase in quantity demanded. In other words, our Engel curves are linear. This is what we demonstrated above.

This property is called **homotheticity**. The lecture slides cover this. Remind me to add this later.



5 Compensated (Hicksian) Demand

5.1 Overview

I am going to rush through this for sake of time. The Hicksian demand is the demand we find when we want to minimize expenditures subject to utility. What does this mean? It means we want to minimize the amount we have to spend in order to reach some fixed utility, U.

The setup is almost the exact same as for Marshallian demand.

1. Find the tangency condition. Set MRS $= \frac{p_x}{p_y}$
2. Solve for either x or y
3. Plug value for x or y into the utility function, $u(x, y)$.
4. Solve for x^c or y^c

5.2 Cobb-Douglas Example

Say we have the following utility function:

$$u = x^2 y$$

Let's just find the compensated demand:

$$MRS = \frac{p_x}{p_y}$$

$$\frac{2y}{x} = \frac{p_x}{p_y}$$

$$y = \frac{xp_x}{2p_y}$$

So now that we have the tangency condition, we have to plug it into our new constraint: the utility function. We know that $u(x, y) = x^2y$, so we will have to replace y.

$$u = (x^2 \frac{xp_x}{2p_y})$$

$$u = (\frac{p_x}{2p_y})x^3$$

$$x^3 = (\frac{2p_y u}{p_x})$$

$$x^c = \sqrt[3]{\frac{2p_y u}{p_x}}$$

And now we have the compensated demand for good x. To find it for y, we do the same thing as before from the tangency condition. I will leave this as an exercise for you, but you should get:

$$y^c = \sqrt[3]{(\frac{2p_y}{p_x})^2 u}$$

5.3 Perfect Compliments Example

Say we have the following utility function:

$$u(x, y) = min3x, y$$

We know that we cannot differentiate this function to find the tangency condition, so what do we do instead? Well, this is where we have to use our economic intuition to get our compensated demand. We know that we consume at the kink point. In other words, our optimal bundle occurs when we equate the inner terms. If $u = min\frac{1}{\alpha}x, \frac{1}{\beta}y$, we must consume where $\frac{1}{\alpha}x = \frac{1}{\beta}y$, otherwise we are inefficiently consuming goods that yield no marginal utility (for further discussion, refer to my optimal choice notes).

So we have that, written out:

$$u = \frac{1}{\alpha}x = \frac{1}{\beta}y$$

This means that our utility is just equal to both terms! Turns out that perfect compliments are much easier in terms of finding compensated demand. Let's refer back to our original problem.

$$u(x, y) = min3x, y$$

$$u = 3x = y$$

Here, we know that we can just solve in terms of x and y.

$$y^c = u$$

and

$$3x = u$$
$$x^c = \frac{u}{3}$$

There we have it! We have just solved for the compensated demand.

5.4 Perfect Substitutes Example

Say we have the following utility function:

$$u(x, y) = 5x + 2y$$

We know that we cannot differentiate this function and solve for a tangency, because these preferences are perfect substitutes. The indifference curves are downward sloping lines, so they will not have points of tangency with the budget constraint. In the case of compensated demand, we are trying to minimize expenditures subject to utility, so we are trying to match the perfect budget constraint subject to utility. This means that we are going to have another corner solution!

How do we find corner solutions for perfect substitutes? We compare marginal utility per dollar! So for this case, we set up the relationship between marginal utilities per dollar.

$$\frac{5}{p_x} = \frac{2}{p_y}$$

So we now have a relationship where we know that the marginal utilities are equal. If $\frac{MUx}{p_x} > \frac{MUy}{p_y}$, we will have a corner solution at the x intercept. But normally, for ordinary demand, this intercept is given by $\frac{I}{p_x}$. Is this still the case for Hicksian (compensated) demand? How has our constraint changed?

Since our constraint is no longer reliant on income, we then have an intercept at $x = \frac{u}{p_x}$. This leads to a very staggering result: if $\frac{MUx}{p_x} > \frac{MUy}{p_y}$, then $x^x = \frac{u}{p_x}$.

Let's solve the given problem with $p_x = 10$. We know that

$$u(x, y) = 5x + 2y$$

And that $p_x = 10$. We can the marginal utilities per dollar to be:

$$\frac{5}{10} = \frac{2}{p_y}$$

We can now see that marginal utility per dollar for good x is just $\frac{1}{2}$. This means that as long as p_y is less than 4, we will only consume good y. Likewise, if $p_y > 4$, we will only consume good x. We then have the following compensated demand:

$$x^c = \begin{cases} \dfrac{u}{p_x} & if p_y > 4, \\ \in [0, \dfrac{u}{p_x}] & if p_y = 4, \\ 0 & if p_y < 4. \end{cases}$$

Et voila. These are the cases for x^c.

5.5 Quasi-linear Example

These functions are going to be kind of brutal. Well, they can be. Let's do an example of a simple quasi-linear.

$$u(x, y) = ln(x) + y$$

$$\frac{\frac{1}{x}}{1} = \frac{p_x}{p_y}$$

$$x = \frac{p_y}{p_x}$$

Throw this into our utility function:

$$u = ln(\frac{p_y}{p_x}) + y$$

Rearranging to solve for y:

$$y^c = u - ln(\frac{p_y}{p_x})$$

However, like with all quasi-linear functions, the value of y could be negative. So we have to see whether y is positive. Setting $y = 0$ we can solve for the edge case:

$$u = ln(\frac{p_y}{p_x})$$

So we conclude that the value of utility must be greater than the natural log of the price ratio in order for the consumer to have $y^c > 0$.